

The Hamiltonian structure of the $N = 2$ supersymmetric GNLS hierarchy

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Abstract

The first two Hamiltonian structures and the recursion operator connecting all evolution systems and Hamiltonian structures of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy are constructed in terms of $N = 2$ superfields in two different superfield bases with local evolution equations. Their bosonic limits are studied in detail. New local and nonlocal bosonic and fermionic integrals both for the $N = 2$ supersymmetric (n, m) -GNLS hierarchy and its bosonic counterparts are derived. As an example, in the $n=1, m=1$ case, the algebra and the symmetry transformations for some of them are worked out and a rich $N=4$ supersymmetry structure is uncovered.

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1. Introduction. Recently there has been an intense research activity on $N=2$ supersymmetric integrable hierarchy. Basically this stems from the interest spurred by the ordinary integrable hierarchies and their relation to the 2D gravity and $N=2$ supersymmetric models representing superstring vacua. Although an analysis of this connection for the $N=2$ supersymmetric integrable hierarchies is still lacking, it is nevertheless a fact that the latter hierarchies have extremely rich and interesting structures. In [1] a large class of such hierarchies was introduced: the $N = 2$ supersymmetric (n, m) Generalized Nonlinear Schrödinger (GNLS) hierarchies. They were subsequently studied in a number of other papers, [2, 3, 4, 5, 6].

The goal of the present letter is to fill a gap in our knowledge of these hierarchies by analyzing their Hamiltonian structures. In particular we produce here the first two Hamiltonian structures and the relevant recursion operators, as well as related local and non-local conserved charges, in two different superfield representations which possess local flow equations.

Let us start with a short summary of the main facts concerning the $N = 2$ supersymmetric (n, m) -GNLS hierarchy [1, 6] which will be useful in what follows.

The Lax operator of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy has the following form¹

$$L = \partial - \frac{1}{2}(F_a \bar{F}_a + F_a \bar{D} \partial^{-1} [D \bar{F}_a]), \quad [D, L] = 0, \quad (1)$$

where $F_a(Z)$ and $\bar{F}_a(Z)$ ($a, b = 1, \dots, n + m$) are chiral and antichiral $N = 2$ superfields

$$D F_a(Z) = 0, \quad \bar{D} \bar{F}_a(Z) = 0, \quad (2)$$

respectively. They are bosonic for $a = 1, \dots, n$ and fermionic for $a = n + 1, \dots, n + m$, i.e., $F_a F_b = (-1)^{d_a d_b} F_b F_a$, where $d_a = 1$ ($d_a = 0$) is the Grassman parity for fermionic (bosonic) superfields; $Z = (z, \theta, \bar{\theta})$ is a coordinate of the $N = 2$ superspace, $dZ \equiv dz d\theta d\bar{\theta}$ and D, \bar{D} are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \quad (3)$$

For $p = 0, 1, 2, \dots$, the Lax operator L provides the consistent flows

$$\frac{\partial}{\partial t_p} L = [(L^p)_{\geq 1}, L]. \quad (4)$$

An infinite number of Hamiltonians can be obtained as follows:

$$H_p = \int dZ \mathcal{H}_p, \quad \mathcal{H}_p \equiv (L^p)_0, \quad (5)$$

where the subscripts ≥ 1 and 0 mean the sum of the purely derivative terms and the constant part of the operator, respectively². The evolution equations (4) for the superfields F_a and \bar{F}_a are local,

$$\frac{\partial}{\partial t_p} F_a = ((L^p)_{\geq 1} F_a)_0, \quad \frac{\partial}{\partial t_p} \bar{F}_a = (-1)^{p+1} ((L^{*p})_{\geq 1} \bar{F}_a)_0, \quad (6)$$

¹Summation over repeated indices is understood and the square brackets mean that the relevant operators act only on superfields inside the brackets.

²An alternative Lax representation of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy was proposed in [3]. Its relation to our Lax representation is not completely clear to us.

and they admit the complex structure

$$F_a^* = (-i)^{d_a-1} \bar{F}_a, \quad \bar{F}_a^* = (-i)^{d_a-1} F_a, \quad \theta^* = \bar{\theta}, \quad \bar{\theta}^* = \theta, \quad t_p^* = (-1)^{p+1} t_p, \quad z^* = z, \quad (7)$$

where i is the imaginary unity, and L^* is the complex-conjugate Lax operator

$$L^* = \partial + \frac{1}{2}(F_a \bar{F}_a + (-1)^{d_a} \bar{F}_a D \partial^{-1} [\bar{D} F_a]), \quad [\bar{D}, L^*] = 0, \quad (8)$$

which also provides consistent flows.

The first three flows from (6) and the first three nontrivial Hamiltonian densities from (5) read as:

$$\begin{aligned} \frac{\partial}{\partial t_0} F_a &= F_a, \quad \frac{\partial}{\partial t_0} \bar{F}_a = -\bar{F}_a; \quad \frac{\partial}{\partial t_1} F_a = F_a', \quad \frac{\partial}{\partial t_1} \bar{F}_a = \bar{F}_a'; \\ \frac{\partial}{\partial t_2} F_a &= F_a'' + D(F_b \bar{F}_b \bar{D} F_a), \quad \frac{\partial}{\partial t_2} \bar{F}_a = -\bar{F}_a'' + \bar{D}(F_b \bar{F}_b D \bar{F}_a), \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{H}_1 &= -\frac{1}{2} F_a \bar{F}_a, \quad \mathcal{H}_2 = \frac{1}{2} (F_a \bar{F}_a' + \frac{1}{4} (F_a \bar{F}_a)^2), \\ \mathcal{H}_3 &= -\frac{1}{2} (F_a \bar{F}_a'' - \frac{1}{2} [\bar{D} F_a \bar{F}_a] [D F_b \bar{F}_b] + F_a \bar{F}_a' F_b \bar{F}_b + \frac{1}{12} (F_a \bar{F}_a)^3), \end{aligned} \quad (10)$$

respectively, where $'$ means the derivative with respect to z . The second flow from the set (9) forms the $N = 2$ supersymmetric (n, m) -GNLS equations.

It is useful, as will be clear in a moment, to introduce an alternative superfield basis by means of $\{J(Z), \bar{\Phi}_j(Z), \Phi_j(Z), j = 1, \dots, l-1, l+1, \dots, n, \dots, n+m\}$,

$$J = \frac{1}{2} (\frac{1}{2} F_a \bar{F}_a - (\ln F_l)'), \quad \bar{\Phi}_j = \frac{1}{\sqrt{2}} \bar{D} (F_l^{-1} F_j), \quad \Phi_j = \frac{1}{\sqrt{2}} D (F_l \bar{F}_j), \quad (11)$$

where the index l is an arbitrary fixed index belonging to the range $1 \leq l \leq n$. A different choice of l leads to different bases, but in the LHS of (11) for simplicity we drop the symbol l . The flows (9) and Hamiltonian densities (10) now become

$$\begin{aligned} \frac{\partial}{\partial t_0} J &= \frac{\partial}{\partial t_0} \bar{\Phi}_i = \frac{\partial}{\partial t_0} \Phi_i = 0; \quad \frac{\partial}{\partial t_1} J = J', \quad \frac{\partial}{\partial t_1} \bar{\Phi}_i = \bar{\Phi}_i', \quad \frac{\partial}{\partial t_1} \Phi_i = \Phi_i'; \\ \frac{\partial}{\partial t_2} J &= (-[D, \bar{D}] J - 2J^2 + \Phi_j \bar{\Phi}_j)', \\ \frac{\partial}{\partial t_2} \Phi_j &= -\Phi_j'' + 4D \bar{D} (J \Phi_j), \quad \frac{\partial}{\partial t_2} \bar{\Phi}_j = \bar{\Phi}_j'' + 4\bar{D} D (J \bar{\Phi}_j), \end{aligned} \quad (12)$$

$$\mathcal{H}_1 = -2J, \quad \mathcal{H}_2 = 2J^2 - \Phi_j \bar{\Phi}_j, \quad \mathcal{H}_3 = \Phi_j' \bar{\Phi}_j + 4J \Phi_j \bar{\Phi}_j - 4\bar{D} J D J - \frac{8}{3} J^3, \quad (13)$$

respectively³. In addition to the first complex structure (7) hidden in this basis, they admit an extra, second complex structure

$$\Phi_j^* = (-i)^{d_j} \bar{\Phi}_j, \quad \bar{\Phi}_j^* = (-i)^{d_j} \Phi_j, \quad J^* = -J, \quad \theta^* = \bar{\theta}, \quad \bar{\theta}^* = \theta, \quad t_p^* = (-1)^{p+1} t_p, \quad z^* = z, \quad (14)$$

which is manifest in this basis, but it is hidden in the former one. Here, d_i is the Grassman parity of the superfields Φ_i and $\bar{\Phi}_i$. We call the basis (11) a KdV-basis, reflecting the fact that for

³Let us recall that Hamiltonian densities are defined up to terms which are fermionic or bosonic total derivatives of arbitrary nonsingular, local functions of the superfields.

$n = 1, m = 0$, eqs. (12) form the flows of the $N = 2$ supersymmetric $a = 4$ KdV hierarchy [7]. In the KdV-basis, the $N = 2$ supersymmetric (n, m) -GNLS hierarchy of integrable equations, together with its Hamiltonians, can be produced using formulas (4), (5), where the Lax operator L (1) is replaced by the gauge related Lax operator

$$L^{KdV} = F_l^{-1} L F_l \equiv \partial - 2J - 2\bar{D}\partial^{-1} \left[D(J - \frac{1}{2}\Phi_j\partial^{-1}\bar{\Phi}_j) \right] + [D\partial^{-1}\bar{\Phi}_j] \bar{D}\partial^{-1}\Phi_j. \quad (15)$$

For general values of the discrete parameters n and m , the Lax representation of the hierarchy corresponding to eqs. (12) was proposed in [4], and its relationships to the $N = 2$ supersymmetric (n, m) -GNLS hierarchy was established in [6]. In addition to the transformations (11), relating eqs. (12) to (9), there are other transformations (for details, see [6]); however, for our purpose here, it will be enough to consider only these transformations.

2. Hamiltonian structure of the $N = 2$ super (n, m) -GNLS hierarchy. A bi-Hamiltonian system of evolution equations can be represented in the following general form:

$$\begin{aligned} \frac{\partial}{\partial t_p} \begin{pmatrix} F_a \\ \bar{F}_a \end{pmatrix} &= (J_1)_{ab} \begin{pmatrix} \delta/\delta F_b \\ \delta/\delta \bar{F}_b \end{pmatrix} H_{p+1} = (J_2)_{ab} \begin{pmatrix} \delta/\delta F_b \\ \delta/\delta \bar{F}_b \end{pmatrix} H_p, \\ (J_1^{(-1)})_{ab} \frac{\partial}{\partial t_p} \begin{pmatrix} F_b \\ \bar{F}_b \end{pmatrix} &= \begin{pmatrix} \delta/\delta F_a \\ \delta/\delta \bar{F}_a \end{pmatrix} H_{p+1}, \end{aligned} \quad (16)$$

where J_1 and J_2 are the first and second Hamiltonian structures. Here we have introduced also the matrix $J_1^{(-1)}$ defined by the relations:

$$J_1 J_1^{(-1)} = \Pi, \quad J_1^{(-1)} J_1 = \bar{\Pi} \quad \Longleftrightarrow \quad \{J_1, J_1^{(-1)}\} = I, \quad (17)$$

where Π ($\bar{\Pi}$)

$$\begin{aligned} \Pi &\equiv - \begin{pmatrix} D\bar{D}\partial^{-1}\delta_{ab}, & 0 \\ 0, & \bar{D}D\partial^{-1}\delta_{ab} \end{pmatrix}, \quad \bar{\Pi} \equiv - \begin{pmatrix} \bar{D}D\partial^{-1}\delta_{ab}, & 0 \\ 0, & D\bar{D}\partial^{-1}\delta_{ab} \end{pmatrix}, \\ \Pi\Pi &= \Pi, \quad \bar{\Pi}\bar{\Pi} = \bar{\Pi}, \quad \Pi\bar{\Pi} = \bar{\Pi}\Pi = 0, \quad \Pi + \bar{\Pi} = I \end{aligned} \quad (18)$$

is the matrix that projects the up and down elements of a column on the chiral (antichiral) and antichiral (chiral) subspaces, respectively. In terms of the Hamiltonian structure J_p , the $N = 2$ supersymmetric Poisson brackets algebra of the superfields F_a and \bar{F}_a are given by the formula:

$$\left\{ \begin{pmatrix} F_a(Z_1) \\ \bar{F}_a(Z_1) \end{pmatrix} \otimes \begin{pmatrix} F_b(Z_2) \\ \bar{F}_b(Z_2) \end{pmatrix} \right\}_p = (J_p)_{ab}(Z_1) \delta^{N=2}(Z_1 - Z_2), \quad (19)$$

where $\delta^{N=2}(Z) \equiv \theta\bar{\theta}\delta(z)$ is the delta function in $N = 2$ superspace and the notation ‘ \otimes ’ stands for the tensor product. In addition to the Jacobi identities and symmetry properties respecting the statistics of the superfields, J_p should also satisfy the chiral consistency conditions

$$J_p \Pi = \bar{\Pi} J_p = 0, \quad J_p \bar{\Pi} = \Pi J_p = J_p, \quad (20)$$

which shows that all the Hamiltonian structures are represented by degenerate matrices. This is the peculiarity of a manifest $N = 2$ superinvariant description of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy in terms of $N = 2$ superfields, which has no analogue in the description

in terms of $N = 1$ superfields or components. One should stress that this is not a pathology of the Hamiltonian structures, but a peculiarity of the $N = 2$ superfield description, which can be easily dealt with.

Using the first three flows of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy and Hamiltonians with densities (10), we have found its first two Hamiltonian structures. The explicit expressions for them as well as for the recursion operator of the hierarchy are presented below. We postpone the discussion of their consistency (the Jacobi identities, the compatibility of the Hamiltonian structures and the hereditariness [8] of the recursion operator) till the end of the next section, where we construct their explicit expressions in the KdV-basis, which are more suitable for this purpose.

In spite of the very complicated form of the first Hamiltonian structure J_1 , the expression for its inverse matrix $J_1^{(-1)}$ is quite simple and looks like⁴

$$(J_1^{(-1)})_{ab} = \frac{1}{4} \Pi \begin{pmatrix} (-1)^{d_b} \bar{F}_a \partial^{-1} \bar{F}_b, & \bar{F}_a \partial^{-1} F_b + 2\delta_{ab} \\ (-1)^{d_a} ((-1)^{d_b} F_a \partial^{-1} \bar{F}_b - 2\delta_{ab}), & (-1)^{d_a} F_a \partial^{-1} F_b \end{pmatrix} \Pi. \quad (21)$$

Actually, in what follows, we need only the explicit expression for the matrix $J_1^{(-1)}$ and due to the very complicated form of J_1 , we do not present it here.

The second Hamiltonian structure has the following form:

$$\begin{aligned} (J_2)_{ab} &= \begin{pmatrix} (J_{11})_{ab}, & (J_{12})_{ab} \\ (J_{21})_{ab}, & (J_{22})_{ab} \end{pmatrix}, \\ (J_{11})_{ab} &= (-1)^{d_a d_b} F_b D \bar{D} \partial^{-1} F_a - F_a D \bar{D} \partial^{-1} F_b, \\ (J_{12})_{ab} &= (-1)^{d_b} (2D \bar{D} - F_c D \bar{D} \partial^{-1} \bar{F}_c) \delta_{ab} + F_a D \bar{D} \partial^{-1} \bar{F}_b, \\ (J_{21})_{ab} &= (2\bar{D} D + (-1)^{d_c} \bar{F}_c \bar{D} D \partial^{-1} F_c) \delta_{ab} - \bar{F}_a \bar{D} D \partial^{-1} F_b, \\ (J_{22})_{ab} &= \bar{F}_a \bar{D} D \partial^{-1} \bar{F}_b - (-1)^{d_a d_b} \bar{F}_b \bar{D} D \partial^{-1} \bar{F}_a. \end{aligned} \quad (22)$$

Knowledge of the first and second Hamiltonian structures allows us to construct the recursion operator R_{ab} of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy using the following general rule:

$$R_{ab} = (J_2 J_1^{(-1)})_{ab} \equiv \Pi \begin{pmatrix} (R_{11})_{ab}, & (R_{12})_{ab} \\ (R_{21})_{ab}, & (R_{22})_{ab} \end{pmatrix} \Pi, \quad \frac{\partial}{\partial t_{p+1}} \begin{pmatrix} F_a \\ \bar{F}_a \end{pmatrix} = R_{ab} \frac{\partial}{\partial t_p} \begin{pmatrix} F_b \\ \bar{F}_b \end{pmatrix}. \quad (23)$$

It is defined up to an arbitrary additive operator which annihilates the column on the r.h.s. of the second relation (23) and can be represented as $C \bar{\Pi}$, where C is an arbitrary matrix-valued pseudo-differential operator. Substituting eqs. (21) and (22) into (23), one can easily obtain the explicit expression for R_{ab} ,

$$\begin{aligned} (R_{11})_{ab} &= (\partial + \frac{1}{2} F_c D \bar{D} \partial^{-1} \bar{F}_c) \delta_{ab} - \frac{1}{2} (-1)^{d_b} (F_a D \bar{D} \partial^{-1} \bar{F}_b + \partial F_a \partial^{-1} \bar{F}_b), \\ (R_{12})_{ab} &= \frac{1}{2} ((-1)^{d_a d_b} F_b D \bar{D} \partial^{-1} F_a - F_a D \bar{D} \partial^{-1} F_b - \partial F_a \partial^{-1} F_b), \\ (R_{21})_{ab} &= \frac{1}{2} (-1)^{d_b} ((-1)^{d_a d_b} \bar{F}_b \bar{D} D \partial^{-1} \bar{F}_a - \bar{F}_a \bar{D} D \partial^{-1} \bar{F}_b - \partial \bar{F}_a \partial^{-1} \bar{F}_b), \\ (R_{22})_{ab} &= (-\partial + \frac{1}{2} (-1)^{d_c} \bar{F}_c \bar{D} D \partial^{-1} F_c) \delta_{ab} - \frac{1}{2} (\bar{F}_a \bar{D} D \partial^{-1} F_b + \partial \bar{F}_a \partial^{-1} F_b), \end{aligned} \quad (24)$$

⁴Hereafter, it is understood that the derivatives ∂ , \bar{D} and D appearing in the Hamiltonian structures, are to be considered as operators that act on whatever is on their right.

and the recurrence relations for the flows,

$$\begin{aligned}\frac{\partial}{\partial t_{p+1}}F_a &= \frac{\partial}{\partial t_p}F_a{}' - \frac{1}{2}(F_a D\bar{D} - D\bar{D}F_a)\partial^{-1}\frac{\partial}{\partial t_p}(F_b\bar{F}_b) + \frac{1}{2}F_b D\bar{D}\partial^{-1}\frac{\partial}{\partial t_p}(\bar{F}_b F_a), \\ \frac{\partial}{\partial t_{p+1}}\bar{F}_a &= -\frac{\partial}{\partial t_p}\bar{F}_a{}' - \frac{1}{2}(\bar{F}_a \bar{D}D - \bar{D}D\bar{F}_a)\partial^{-1}\frac{\partial}{\partial t_p}(F_b\bar{F}_b) + \frac{(-1)^{d_b}}{2}\bar{F}_b \bar{D}D\partial^{-1}\frac{\partial}{\partial t_p}(F_b\bar{F}_a).\end{aligned}\quad (25)$$

At this point let us make a remark which will be useful in the following. Taking into account the local nature of flows (6) of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy, a simple inspection of the recurrence relations (25) allows one to conclude that the time derivatives of the superfunctions

$$\mathcal{H}_{1,ab} \equiv F_a \bar{F}_b \quad (26)$$

should be represented as the sum of total bosonic and fermionic derivatives of some local superfield functions. Moreover, the evolution equations for the function $\mathcal{H}_{1,aa}$ should contain only a total bosonic derivative. In other words the quantities

$$H_{1,ab} = \int dZ \mathcal{H}_{1,ab}, \quad \widetilde{H}_0 = \int dz \mathcal{H}_{1,aa} \quad (27)$$

are to be integrals of the flows. For the flows (9), this can be checked by simple direct calculations. For the p -th flow of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy, the evolution equation for the $\mathcal{H}_{1,aa}$ takes the following general form [6]:

$$-\frac{1}{2}\frac{\partial}{\partial t_p}\mathcal{H}_{1,aa} = ((L^p)_0)' \quad (28)$$

which agrees with the above-mentioned arguments.

Using the Poisson brackets algebra (19), (22) one can calculate the Poisson brackets between the integrals $H_{1,ab}$ and \widetilde{H}_0 (27)

$$\{H_{1,ab}, \widetilde{H}_0\} = 0, \quad \{H_{1,ab}, H_{1,cd}\} = 2(-1)^{d_a d_d} \delta_{bc} H_{2,da} - 2(-1)^{d_d(d_c+d_b+1)} \delta_{ad} H_{2,bc}, \quad (29)$$

where the new nonlocal integrals

$$H_{2,ab} = \int dZ \bar{F}_a L F_b, \quad (30)$$

have been introduced, L being the Lax operator (1). The integrals complex-conjugate with respect to complex structure (7) $H_{2,ad}^*$ are related to $H_{2,da}$ as $H_{2,ad}^* = (-1)^{\frac{(d_a+d_d)^2}{2}} H_{2,da}$. Repeatedly applying the same procedure one can generate new series of nonlocal integrals.

Acting p -times with the recursion operator (23), (24) on the zeroth flow from the set (9) and on the second Hamiltonian structure (22) of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy, one can derive p -th flow, as well as the $(p+2)$ -th Hamiltonian structure,

$$\frac{\partial}{\partial t_p} \begin{pmatrix} F_a \\ \bar{F}_a \end{pmatrix} = (R^p)_{ab} \begin{pmatrix} F_b \\ -\bar{F}_b \end{pmatrix}, \quad J_{p+2} = R^p J_2, \quad (31)$$

respectively. Substituting the explicit expressions (23), (24) for the recursion operator into the first formula of eqs. (31), we obtain, for example, the following set of equations for the 3-th flow:

$$\begin{aligned}\frac{\partial}{\partial t_3}F_a &= F_a''' + \frac{3}{2}D(\bar{D}(F_b\bar{F}_b F_a{}') - \frac{1}{2}(F_b\bar{F}_b)^2 \bar{D}F_a + [D\bar{F}_b][\bar{D}F_b]\bar{D}F_a), \\ \frac{\partial}{\partial t_3}\bar{F}_a &= \bar{F}_a''' - \frac{3}{2}\bar{D}(D(F_b\bar{F}_b \bar{F}_a{}') + \frac{1}{2}(F_b\bar{F}_b)^2 D\bar{F}_a - [D\bar{F}_b][\bar{D}F_b]D\bar{F}_a),\end{aligned}\quad (32)$$

which coincides with the corresponding set that can be derived using eqs. (6) and gives a confirmation of the above-constructed formulas. The flows allow the Hamiltonian densities (5) corresponding to them to be constructed using eq. (28),

$$\mathcal{H}_p = -\frac{1}{2}\partial^{-1}\frac{\partial}{\partial t_p}(F_b\bar{F}_b), \quad (33)$$

without knowing the Lax operator. Thus, almost all information about the $N = 2$ supersymmetric (n, m) -GNLS hierarchy is encoded in its recursion operator.

For the particular cases $n = 0, m = 1$ and $n = 1, m = 0$, the expressions (21), (22) and (23)-(24) for the Hamiltonian structures and the recursion operator of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy reproduce the corresponding expressions constructed in [1, 5].

3. Hamiltonian structure of the $N = 2$ super (n, m) -GNLS hierarchy in the KdV-basis. In the KdV-basis (11), the general set of bi-Hamiltonian equations (16) takes the form:

$$\begin{aligned} \frac{\partial}{\partial t_p} \begin{pmatrix} J \\ \bar{\Phi}_i \\ \Phi_i \end{pmatrix} &= (J_1^{KdV})_{ij} \begin{pmatrix} \delta/\delta J \\ \delta/\delta \bar{\Phi}_j \\ \delta/\delta \Phi_j \end{pmatrix} H_{p+1} = (J_2^{KdV})_{ij} \begin{pmatrix} \delta/\delta J \\ \delta/\delta \bar{\Phi}_j \\ \delta/\delta \Phi_j \end{pmatrix} H_p, \\ (J_1^{KdV})_{ij}^{(-1)} \frac{\partial}{\partial t_p} \begin{pmatrix} J \\ \bar{\Phi}_j \\ \Phi_j \end{pmatrix} &= \begin{pmatrix} \delta/\delta J \\ \delta/\delta \bar{\Phi}_j \\ \delta/\delta \Phi_j \end{pmatrix} H_{p+1}. \end{aligned} \quad (34)$$

The Hamiltonian structures J_p^{KdV} are related to J_p (31) by the general rule⁵

$$(J_p^{KdV})_{ij} = \mathcal{G}_{ia}(J_p)_{ab}(\mathcal{G}^T)_{bj}, \quad (J_p^{(-1)})_{ab} = (\mathcal{G}^T)_{ai}(J_p^{KdV})_{ij}^{(-1)}\mathcal{G}_{jb}, \quad (35)$$

where⁶

$$\mathcal{G}_{ia} \equiv \begin{pmatrix} \frac{1}{2}(\frac{1}{2}(-1)^{d_a}\bar{F}_a - \partial F_l^{-1}\delta_{al}), & \frac{1}{4}F_a \\ \frac{1}{\sqrt{2}}\bar{D}F_l^{-1}(\delta_{ia} - F_l^{-1}F_i\delta_{al}), & 0 \\ \frac{1}{\sqrt{2}}D\bar{F}_i\delta_{al}, & \frac{1}{\sqrt{2}}DF_l\delta_{ia} \end{pmatrix} \Pi \quad (36)$$

is the matrix of Fréchet derivatives corresponding to the transformation $\{J, \bar{\Phi}_i, \Phi_i\} \Rightarrow \{F_a, \bar{F}_a\}$ (11) to the KdV-basis. Using eqs. (21), (22) and (35), one can derive the following expressions for the first⁷,

$$\begin{aligned} (J_1^{KdV})_{ij}^{(-1)} &= \begin{pmatrix} 4, & 0, & 0 \\ 0, & 0, & (-1)^{d_i}D\bar{D}\partial^{-1}\delta_{ij} \\ 0, & \bar{D}D\partial^{-1}\delta_{ij}, & 0 \end{pmatrix} \partial^{-1}, \\ (J_1^{KdV})_{ij} &= \begin{pmatrix} \frac{1}{4}, & 0, & 0 \\ 0, & 0, & \bar{D}D\partial^{-1}\delta_{ij} \\ 0, & (-1)^{d_i}D\bar{D}\partial^{-1}\delta_{ij}, & 0 \end{pmatrix} \partial, \\ J_1^{KdV}(J_1^{KdV})^{(-1)} &= \begin{pmatrix} 1, & 0 \\ 0, & \bar{\Pi} \end{pmatrix}, \quad (J_1^{KdV})^{(-1)}J_1^{KdV} = \begin{pmatrix} 1, & 0 \\ 0, & \Pi \end{pmatrix}, \end{aligned} \quad (37)$$

⁵Let us recall the rules for the adjoint conjugation operation T : $D^T = -D$, $\bar{D}^T = -\bar{D}$, $(QP)^T = (-1)^{d_Q d_P} P^T Q^T$, where Q and P are arbitrary operators. In addition, for matrices, it is necessary to take the operation of the matrix transposition. All other rules can be derived using these.

⁶Let us remember that the index l is an arbitrary fixed index belonging to the range $1 \leq l \leq n$. Therefore, in (36), there is no summation over repeated indices l .

⁷Here, d_i is the Grassman parity of the superfields Φ_i and $\bar{\Phi}_i$.

and for the second,

$$\begin{aligned}
(J_2^{KdV})_{ij} &= \begin{pmatrix} J_{11}, & (J_{12})_j, & (J_{13})_j \\ (J_{21})_i, & (J_{22})_{ij}, & (J_{23})_{ij} \\ (J_{31})_i, & (J_{32})_{ij}, & (J_{33})_{ij} \end{pmatrix}, \\
J_{11} &= -\frac{1}{2}([D, \bar{D}] \partial + \bar{D} J D + D J \bar{D} + \partial J + J \partial), \\
(J_{12})_j &= \frac{1}{2}(\bar{\Phi}_j D + (-1)^{d_j} D \bar{\Phi}_j) \bar{D}, \quad (J_{13})_j = \frac{1}{2}(\Phi_j \bar{D} + (-1)^{d_j} \bar{D} \Phi_j) D, \\
(J_{21})_i &= \frac{1}{2} \bar{D}((-1)^{d_i} \bar{\Phi}_i D + D \bar{\Phi}_i), \quad (J_{31})_i = \frac{1}{2} D((-1)^{d_i} \Phi_i \bar{D} + \bar{D} \Phi_i), \\
(J_{22})_{ij} &= \bar{\Phi}_i D \bar{D} \partial^{-1} \bar{\Phi}_j - (-1)^{d_i d_j} \bar{\Phi}_j D \bar{D} \partial^{-1} \bar{\Phi}_i, \\
(J_{33})_{ij} &= -\Phi_i \bar{D} D \partial^{-1} \bar{\Phi}_j + (-1)^{d_i d_j} \Phi_j \bar{D} D \partial^{-1} \Phi_i, \\
(J_{23})_{ij} &= (\bar{D}(\partial - 2J)D - (-1)^{d_m} \bar{\Phi}_m \bar{D} D \partial^{-1} \Phi_m) \delta_{ij} + \bar{\Phi}_i \bar{D} D \partial^{-1} \Phi_j, \\
(J_{32})_{ij} &= -(-1)^{d_j} (D(\partial + 2J) \bar{D} - \Phi_m D \bar{D} \partial^{-1} \bar{\Phi}_m) \delta_{ij} - \Phi_i D \bar{D} \partial^{-1} \bar{\Phi}_j,
\end{aligned} \tag{38}$$

Hamiltonian structures, respectively, and construct the recursion operator:

$$\begin{aligned}
R_{ij}^{KdV} &= (J_2^{KdV} (J_1^{KdV})^{(-1)})_{ij} \equiv \begin{pmatrix} 4J_{11}, & -(J_{13})_j, & -(-1)^{d_j} (J_{12})_j \\ 4(J_{21})_i, & -(J_{23})_{ij}, & -(-1)^{d_j} (J_{22})_{ij} \\ 4(J_{31})_i, & -(J_{33})_{ij}, & -(-1)^{d_j} (J_{32})_{ij} \end{pmatrix} \partial^{-1}, \\
\frac{\partial}{\partial t_{p+1}} \begin{pmatrix} J \\ \bar{\Phi}_i \\ \Phi_i \end{pmatrix} &= R_{ij}^{KdV} \frac{\partial}{\partial t_p} \begin{pmatrix} J \\ \bar{\Phi}_j \\ \Phi_j \end{pmatrix} = (R^{KdV})_{ij}^p \begin{pmatrix} J' \\ \bar{\Phi}_j' \\ \Phi_j' \end{pmatrix}
\end{aligned} \tag{39}$$

of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy in the KdV-basis.

The Jacobi identities for the first Hamiltonian structure J_1^{KdV} (37) are obviously satisfied as for the constant-coefficient operator with the correct symmetry properties. For the particular cases $n = 1, m = 0$ ($n = 0, m = 1$) and $n = 1, m = 1$, J_1^{KdV} was found in [10, 7] and [9], respectively, and the hereditary recursion operator for the former case was constructed in [11].

In regard to the Jacobi identities for the second Hamiltonian structure (38), for the particular case $n = 1, m = 0$ ($n = 0, m = 1$), J_2^{KdV} coincides with the $N = 2$ superconformal algebra which is the second Hamiltonian structure of the $N = 2$ $a = 4$ KdV hierarchy [7], and for the case $n = 1, m = 1$, it forms the $N = 4$ $SU(2)$ superconformal algebra—the second Hamiltonian structure of the $N = 4$ $SU(2)$ -KdV hierarchy [9]. Therefore, for these cases, they are satisfied. We did not check them for the other values of the discrete parameters n and m . However, we have verified the J_2^{KdV} for the first four flows of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy at arbitrary values of n and m . Moreover, in what follows, we check that in the bosonic limit it correctly reproduces the second Hamiltonian structure of the bosonic GNLS hierarchy for arbitrary value of the parameter m . Taking into account these arguments, it is natural to expect that the expressions (38) for the general supersymmetric case are correct, but we do not present a proof here.

For arbitrary values of the parameters n and m , the Hamiltonian structures J_1^{KdV} and J_2^{KdV} are obviously compatible: the deformation of the superfield $J \Rightarrow J + \gamma$, where γ is an arbitrary parameter, transforms J_2^{KdV} into the Hamiltonian structure defined by their algebraic sum $J_2^{KdV} - 2\gamma J_1^{KdV}$. Thus, one can conclude that the recursion operator R_{ij}^{KdV} (39) is hereditary as the operator obtained from the compatible pair of the Hamiltonian structures [8].

Let us remark that the second Hamiltonian structures J_2^{KdV} form the extended $N = 2$ superconformal algebras, possessing a manifest $N = 2$ supersymmetry, with the $N = 2$ stress-tensor $J(Z)$ and spin-1 primary fermionic and bosonic supercurrents, $\Phi_i(Z)$ and $\bar{\Phi}_i(Z)$. For general values of the parameters n and m , these algebras are nonlocal. Taking into account that the $N = 2$ superconformal algebra can be derived via the Hamiltonian reduction of the $N = 2$ $sl(2|1)$ affine superalgebra, it is reasonable to conjecture the existence of a similar relation of our superalgebras to the $N = 2$ $sl(k|k-1)$ affine superalgebras [12]. The detailed analysis of this complicated problem is however out the scope of the present letter.

In the KdV-basis (11), there are also series of local and nonlocal additional integrals,

$$\begin{aligned} H_0^{KdV} &= \int dz J, \quad \widetilde{H}_{1,i}^{KdV} = \int dz \bar{D}\Phi_i, \\ H_{1,ij}^{KdV} &= \int dZ \Phi_i \partial^{-1} \bar{\Phi}_j, \quad H_{1,i}^{KdV} = \int dZ \left[J - \frac{1}{2} \Phi_j \partial^{-1} \bar{\Phi}_j \right] D \partial^{-1} \bar{\Phi}_i, \\ \widetilde{H}_2^{KdV} &= \int dZ \left[J - \frac{1}{2} \Phi_j \partial^{-1} \bar{\Phi}_j \right] L^{KdV} 1, \quad \widetilde{H}_{2,i}^{KdV} = \int dz [\bar{D}\Phi_i] L^{KdV} 1, \\ H_{2,ij}^{KdV} &= \int dZ \Phi_i L^{KdV} \partial^{-1} \bar{\Phi}_j, \quad H_{2,i}^{KdV} = \int dZ \left[J - \frac{1}{2} \Phi_j \partial^{-1} \bar{\Phi}_j \right] L^{KdV} D \partial^{-1} \bar{\Phi}_i, \end{aligned} \quad (40)$$

corresponding to the integrals (26), (27) and (30), as well as their complex-conjugates with respect to complex structure (14). Up to normalization constants, here are some of them:

$$\begin{aligned} \widetilde{H}_{1,i}^{*KdV} &= \int dz D \bar{\Phi}_i, \quad H_{1,i}^{*KdV} = \int dZ \left[J + \frac{(-1)^{d_j}}{2} \bar{\Phi}_j \partial^{-1} \Phi_j \right] \bar{D} \partial^{-1} \Phi_i, \\ \widetilde{H}_{2,i}^{*KdV} &= \int dz [D \bar{\Phi}_i] L^{*KdV} 1, \quad H_{2,i}^{*KdV} = \int dZ \left[J + \frac{(-1)^{d_j}}{2} \bar{\Phi}_j \partial^{-1} \Phi_j \right] L^{*KdV} \bar{D} \partial^{-1} \Phi_i. \end{aligned} \quad (41)$$

These are algebraically independent with respect to integrals (40). Here, L^{KdV} is the Lax operator (15), and L^{*KdV} is its complex-conjugate operator with respect to complex structure (14).

Let us remark that for the particular case $n = 1, m = 1$, the Poisson brackets between the superfield integrals H_0^{KdV} , $\widetilde{H}_{1,1}^{KdV}$ and $\widetilde{H}_{1,1}^{*KdV}$, calculated using the second Hamiltonian structure J_2^{KdV} , form the global $N = 4$ supersymmetric algebra in one dimension. The Poisson brackets between these integrals and the superfields J , Φ_1 and $\bar{\Phi}_1$ generate the $N = 4$ infinitesimal transformations of the last ones, which are symmetry transformations of the $N = 4$ $SU(2)$ -KdV hierarchy. As an example, we present the transformations generated by the sum $\bar{\epsilon} \widetilde{H}_{1,1}^{KdV} + \epsilon \widetilde{H}_{1,1}^{*KdV}$ of the integrals,

$$\delta J = \frac{1}{2}(\epsilon D \bar{\Phi}_1 + \bar{\epsilon} \bar{D} \Phi_1), \quad \delta \Phi_1 = -2\epsilon D J, \quad \epsilon \bar{\Phi}_1 = -2\bar{\epsilon} \bar{D} J, \quad (42)$$

which coincide with the transformations of the hidden $N = 2$ supersymmetry of the $N = 4$ $SU(2)$ -KdV hierarchy, derived in [9]. Here, ϵ and $\bar{\epsilon}$ are the fermionic parameters of the transformation. In the former superfield basis $\{F_a, \bar{F}_a\}$, the $N = 4$ supersymmetric transformations are generated by the integrals \widetilde{H}_0 and $H_{1,12}$ (26), (27) as well as by the integral

$$\widetilde{H}_{1,1}^* = \int dZ \frac{F_2}{F_1}, \quad (43)$$

which corresponds to the integral $\widetilde{H}_{1,1}^{*KdV}$. In this basis the transformations (42) become⁸

$$\begin{aligned}\delta F_a &= \frac{(-1)^{d_a}}{\sqrt{2}}(\bar{\epsilon}((2\partial + F_b D \bar{D} \partial^{-1} \bar{F}_b) F_1 \delta_{a2} + F_1 D \bar{D} \partial^{-1} F_a \bar{F}_2) + \epsilon F_2 \delta_{a1}), \\ \delta \bar{F}_a &= \frac{(-1)^{d_a}}{\sqrt{2}}(\bar{\epsilon}((-2\partial + (-1)^{d_b} \bar{F}_b \bar{D} D \partial^{-1} F_b) \bar{F}_2 \delta_{a1} - (-1)^{d_a} \bar{F}_2 \bar{D} D \partial^{-1} \bar{F}_a F_1) - \epsilon \bar{F}_1 \delta_{a2}).\end{aligned}\quad (44)$$

We have checked that transformations (44) are indeed the symmetry transformations for the $N = 2$ supersymmetric $(1, 1)$ -GNLS equations (9) and that their Lie brackets coincide with the brackets for the transformations (42). Thus, the $N = 2$ supersymmetric $(1, 1)$ -GNLS hierarchy can also be called the $N = 4$ supersymmetric NLS-mNLS hierarchy reflecting the name of its first nontrivial bosonic representative (see the next section)⁹. In fact, it possesses one more $N = 4$ supersymmetry which one can derive by the complex-conjugation operation with respect to the first complex structure (7) applied either to the integrals \bar{H}_0 , $H_{1,12}$ and $\bar{H}_{1,1}^*$ or directly to their $N = 4$ supersymmetry transformations. One can easily observe that under this operation the first integral, which form the standard $N = 2$ supersymmetry, only change the overall sign, while the other two integrals are drastically changed (together with the transformations generated by them). Thus, one can conclude that these two different $N = 4$ supersymmetries intersect along the $N = 2$ supersymmetry. Without going to more details, we present the new transformations,

$$\begin{aligned}\delta F_a &= \frac{(-1)^{d_a}}{\sqrt{2}}(\tilde{\epsilon}((2\partial + F_b D \bar{D} \partial^{-1} \bar{F}_b) F_2 \delta_{a1} - (-1)^{d_a} F_2 D \bar{D} \partial^{-1} F_a \bar{F}_1) - \bar{\tilde{\epsilon}} F_1 \delta_{a2}), \\ \delta \bar{F}_a &= \frac{(-1)^{d_a}}{\sqrt{2}}(\tilde{\epsilon}((2\partial - (-1)^{d_a} \bar{F}_b \bar{D} D \partial^{-1} F_b) \bar{F}_1 \delta_{a2} + \bar{F}_1 \bar{D} D \partial^{-1} \bar{F}_a F_2) - \bar{\tilde{\epsilon}} \bar{F}_2 \delta_{a1}),\end{aligned}\quad (45)$$

which are the counterparts of the transformations (44). Here, $\tilde{\epsilon}$ and $\bar{\tilde{\epsilon}}$ are the two new independent fermionic parameters. We have also checked that the algebraic closure of these two sets of $N = 4$ supersymmetric generator integrals contain new integrals, but the detailed analysis of the resulting algebra will be discussed elsewhere. Let us only mention that the transformations with the parameters ϵ and $\bar{\tilde{\epsilon}}$ in the closing generate the transformations of the $GL(1|1)$ supergroup¹⁰.

As for generic values of the parameters n and m , the algebras of the corresponding integrals and the transformation properties of the superfields can be derived in a similar way, but again their detailed description will not be given here.

4. Bosonic limit of the $N = 2$ super (n, m) -GNLS Hamiltonian structure. To derive the bosonic limit, we set all fermionic components of the superfields F_a and \bar{F}_a equal to zero and define the bosonic components as [1]

$$b_\alpha = \frac{1}{\sqrt{2}} F_\alpha|, \quad \bar{b}_\beta = \frac{1}{\sqrt{2}} \bar{F}_\beta|, \quad 1 \leq \alpha, \beta \leq n,$$

⁸To derive the transformations generated by integral (43), it is necessary to remove the ambiguity in the operators $\bar{D} D \partial^{-1} 1$ and $\partial \partial^{-1} 1$ that appear in the calculations by setting $\bar{D} D \partial^{-1} 1 = (\bar{D} D \partial^{-1}) 1 \equiv -1$ and $\partial \partial^{-1} 1 = (\partial \partial^{-1}) 1 \equiv 1$. Let us remark that in spite of the chiral nature of the integrated function $\frac{F_2}{F_1}$, in general, the integral (43) is not equal to zero due to its singularity, and the surface terms should be taken into account.

⁹For other examples of $N = 4$ supersymmetric NLS-type integrable hierarchies, see the recent paper [13]

¹⁰Let us recall that, for general values of the parameters n and m , the $N = 2$ supersymmetric (n, m) -GNLS hierarchy is invariant with respect to $GL(n|m)$ supergroup [1].

$$g_s = \frac{1}{\sqrt{2}} \overline{D} F_{s+n} | \exp(-\partial^{-1}(b_\beta \bar{b}_\beta)), \quad \bar{g}_p = \frac{1}{\sqrt{2}} D \bar{F}_{p+n} | \exp(\partial^{-1}(b_\beta \bar{b}_\beta)), \quad 1 \leq s, p \leq m, \quad (46)$$

where $|$ means the $(\theta, \bar{\theta}) \rightarrow 0$ limit. In terms of such components, the second flow equations (9) for the fields b_α, \bar{b}_α and g_s, \bar{g}_s are completely decoupled:

$$\frac{\partial}{\partial t_2} b_\alpha = b_\alpha'' - 2b_\beta \bar{b}_\beta b_\alpha', \quad \frac{\partial}{\partial t_2} \bar{b}_\alpha = -\bar{b}_\alpha'' - 2b_\beta \bar{b}_\beta \bar{b}_\alpha', \quad (47)$$

$$\frac{\partial}{\partial t_2} g_s = g_s'' - 2g_p \bar{g}_p g_s, \quad \frac{\partial}{\partial t_2} \bar{g}_s = -\bar{g}_s'' + 2g_p \bar{g}_p \bar{g}_s. \quad (48)$$

The set of equations (48) and (47) form the bosonic GNLS [14] and modified GNLS (mGNLS) [1] equations, respectively.

The bosonic limit of the Hamiltonian structures (21) and (22), recursion operator (23), (24), and integrals (26), (27) and (30), corresponding to equations (47) and (48), also splits into two independent structures, which one can see from the following explicit expressions:

$$(J_1^{mGNLS})_{\alpha\beta}^{-1} = \begin{pmatrix} \bar{b}_\alpha' \partial^{-1} \bar{b}_\beta + \bar{b}_\alpha \partial^{-1} \bar{b}_\beta', & \bar{b}_\alpha' \partial^{-1} b_\beta + \bar{b}_\alpha \partial^{-1} b_\beta \partial + \partial \delta_{\alpha\beta} \\ -b_\alpha' \partial^{-1} \bar{b}_\beta - b_\alpha \partial^{-1} \bar{b}_\beta \partial + \partial \delta_{\alpha\beta}, & -b_\alpha' \partial^{-1} b_\beta - b_\alpha \partial^{-1} b_\beta' \end{pmatrix}, \quad (49)$$

$$(J_1^{GNLS})_{sp}^{-1} = \begin{pmatrix} 0, & -\delta_{sp} \\ \delta_{sp}, & 0 \end{pmatrix}, \quad (J_1^{GNLS})_{sp} = \begin{pmatrix} 0, & \delta_{sp} \\ -\delta_{sp}, & 0 \end{pmatrix}, \quad (50)$$

for the first, and,

$$(J_2^{mGNLS})_{\alpha\beta} = \begin{pmatrix} b_\beta \partial^{-1} b_\alpha - b_\alpha \partial^{-1} b_\beta, & (1 - b_\gamma \partial^{-1} \bar{b}_\gamma) \delta_{\alpha\beta} + b_\alpha \partial^{-1} \bar{b}_\beta \\ -(1 + \bar{b}_\gamma \partial^{-1} b_\gamma) \delta_{\alpha\beta} + \bar{b}_\alpha \partial^{-1} b_\beta, & \bar{b}_\beta \partial^{-1} \bar{b}_\alpha - \bar{b}_\alpha \partial^{-1} \bar{b}_\beta \end{pmatrix}, \quad (51)$$

$$(J_2^{GNLS})_{sp} = \begin{pmatrix} g_p \partial^{-1} g_s + g_s \partial^{-1} g_p, & (\partial - g_c \partial^{-1} \bar{g}_c) \delta_{sp} - g_s \partial^{-1} \bar{g}_p \\ (\partial - \bar{g}_c \partial^{-1} g_c) \delta_{sp} - \bar{g}_s \partial^{-1} g_p, & \bar{g}_p \partial^{-1} \bar{g}_s + \bar{g}_s \partial^{-1} \bar{g}_p \end{pmatrix}, \quad (52)$$

for the second Hamiltonian structures, and,

$$R_{\alpha\beta}^{mGNLS} = \begin{pmatrix} (1 - b_\gamma \partial^{-1} \bar{b}_\gamma) \partial \delta_{\alpha\beta} - b_\alpha' \partial^{-1} \bar{b}_\beta, & b_{[\alpha} \partial^{-1} b_{\beta]} + [b_\alpha \partial^{-1} b_\beta, \partial] \\ \bar{b}_{[\alpha} \partial^{-1} \bar{b}_{\beta]} + [\bar{b}_\alpha \partial^{-1} \bar{b}_\beta, \partial], & -(1 + \bar{b}_\gamma \partial^{-1} b_\gamma) \partial \delta_{\alpha\beta} - \bar{b}_\alpha' \partial^{-1} b_\beta \end{pmatrix}, \quad (53)$$

$$R_{sp}^{GNLS} = \begin{pmatrix} (\partial - g_c \partial^{-1} \bar{g}_c) \delta_{sp} - g_s \partial^{-1} \bar{g}_p, & -g_p \partial^{-1} g_s - g_s \partial^{-1} g_p \\ \bar{g}_p \partial^{-1} \bar{g}_s + \bar{g}_s \partial^{-1} \bar{g}_p, & (-\partial + \bar{g}_c \partial^{-1} g_c) \delta_{sp} + \bar{g}_s \partial^{-1} g_p \end{pmatrix}, \quad (54)$$

for the recursion operator, as well as,

$$\begin{aligned} H_0^{mGNLS} &= \int dz b_\alpha \bar{b}_\alpha, & H_{1,\alpha\beta}^{mGNLS} &= \int dz b_\alpha \bar{b}_\beta', \\ H_{2,\alpha\beta}^{mGNLS} &= \int dz \bar{b}_\alpha \partial (1 - b_\gamma \partial^{-1} \bar{b}_\gamma) b_\beta' \equiv \int dz \bar{b}_\alpha L^{mGNLS} b_\beta', \end{aligned} \quad (55)$$

$$H_{1,sp}^{GNLS} = \int dz g_s \bar{g}_p, \quad H_{2,sp}^{GNLS} = \int dz \bar{g}_s (\partial - g_c \partial^{-1} \bar{g}_c) g_p \equiv \int dz \bar{g}_s L^{GNLS} g_p, \quad (56)$$

for the integrals, where L^{mGNLS} and L^{GNLS} are the Lax operators of the mGNLS and GNLS hierarchies¹¹. In the bosonic limit, the expressions for the Hamiltonian densities (5) given by eq. (33) look as follows:

$$\mathcal{H}_p^{mGNLS} = \partial^{-1} \frac{\partial}{\partial t_p} (b_\alpha \bar{b}_\alpha'), \quad (57)$$

$$\mathcal{H}_p^{GNLS} = \partial^{-1} \frac{\partial}{\partial t_p} (g_s \bar{g}_s). \quad (58)$$

The Hamiltonian structures (50), (52) and (58), as well as the recursion operator (54) of the GNLS hierarchy, reproduce the corresponding expressions constructed in [15]. Regarding Hamiltonian structures (49) and (51), as well as recursion operator (53) for the mGNLS hierarchy, they coincide for the particular case $n = 1$ with the corresponding expressions obtained in [16]. However, for a general value of n , to our knowledge, they are presented for the first time.

5. Conclusion. In this Letter, we have constructed the first and second Hamiltonian structures, (21), (22), (37) and (38), as well as the recursion operators, (23), (24) and (39), which connect all evolution systems and Hamiltonian structures of the $N = 2$ supersymmetric (n, m) -GNLS hierarchy in two different superfield bases characterized by local evolution equations. For general values of n and m , to our knowledge, they are presented here for the first time. We have also produced their bosonic counterparts (49)–(54). Finally we have constructed the new local and nonlocal bosonic and fermionic integrals (26), (27), (30), (40), (41), (43), (55) and (56) of the supersymmetric and bosonic hierarchies.

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¹¹The operator L^{mGNLS} is gauge-related to the Lax operator \tilde{L}_2 proposed in [1] for the mGNLS hierarchy, $L^{mGNLS} = G^{-1} \tilde{L}_2 G$, where $G \equiv \exp(-\partial^{-1}(b_\beta \bar{b}_\beta))$.

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